

DUALITY AND UNIVERSALITY FOR STABLE PAIR INVARIANTS OF SURFACES

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ABSTRACT. Let β be a curve class on a surface S . The moduli space of stable pairs on S with class β carries a natural reduced virtual cycle [KT1, KT2]. This cycle is defined when $h^2(L) = 0$ for any *effective* $L \in \text{Pic}^\beta(S)$ (weak assumption). When $h^2(L) = 0$ for *any* $L \in \text{Pic}^\beta(S)$ (strong assumption), the associated invariants are given by universal functions in β^2 , $\beta \cdot c_1(S)$, $c_1(S)^2$, $c_2(S)$, and certain invariants of the ring structure of $H^*(S, \mathbb{Z})$.

In this paper, we show the following. (1) Universality *no longer* holds when just the weak assumption is satisfied. (2) For any S, β (no conditions), the BPS spectrum of the non-reduced stable pair invariants of S, β with maximal number of point insertions consists of a single number. This number is the Seiberg-Witten invariant of S, β . (3) The GW/PT correspondence for $X = K_S$ implies Taubes' GW/SW correspondence in certain cases, e.g. when β is irreducible. (4) When $p_g(S) = 0$, the difference between the stable pair invariants in class β and $K_S - \beta$ is universal.

1. INTRODUCTION

In [PT1], R. Pandharipande and R. P. Thomas introduce stable pairs on projective 3-folds X and show their moduli space is a component of the moduli space of all complexes in the bounded derived category $D^b(X)$. Formally, a stable pair (F, s) on X consists of a pure dimension 1 sheaf F on X and a section $s \in H^0(F)$ with 0-dimensional cokernel. The moduli space of stable pairs has a perfect obstruction theory, which is symmetric in the case X is Calabi-Yau. The associated invariants are known as stable pair invariants and are closely related to the Donaldson-Thomas and Gromov-Witten invariants of X [Bri, MNOP1, MNOP2, MOOP, PP1, PP2, PT1, PT2, Tod].

We consider the case $X = K_S$ is the total space of the canonical bundle over a smooth projective surface S . Let $P_\chi(X, \beta)$ denote the moduli space of stable pairs (F, s) on X with class $\beta \in H^2(S, \mathbb{Z})$ and $\chi(F) = \chi$. The space $P_\chi(X, \beta)$ carries a perfect obstruction theory but need not be compact. Using the \mathbb{C}^* -action on the fibres of X , we get an induced perfect obstruction theory on the components of $P_\chi(X, \beta)^{\mathbb{C}^*}$. These components are compact and the stable pair invariants of X are *defined* by integration over the virtual cycles of

these components [GP]. The easiest component of $P_\chi(X, \beta)^{\mathbb{C}^*}$ consists of stable pairs scheme theoretically supported on the zero section $S \subset X$, i.e. $P_\chi(S, \beta)$. Denote the Hilbert scheme of curves on S with class β by $H_\beta := \text{Hilb}_\beta(S)$ and the universal curve by $\mathcal{C} \rightarrow H_\beta$. Let n be determined by $\chi = 1 - h + n$, where h is the arithmetic genus of curves in H_β

$$2h - 2 = \beta(\beta + k), \quad k := c_1(\mathcal{O}(K_S)) \in H^2(S, \mathbb{Z}).$$

Given a stable pair $[s : \mathcal{O}_S \rightarrow F]$ on S , the scheme theoretic support of F is a Gorenstein curve $C \subset S$. The cokernel Q of s gives rise to a 0-dimensional closed subscheme $Z \subset C$ via the surjection $\mathcal{O}_C \rightarrow \mathcal{E}xt^1(Q, \mathcal{O}_C)$ obtained from dualizing. This provides an isomorphism [PT3]

$$P_\chi(S, \beta) \cong \text{Hilb}^n(\mathcal{C}/H_\beta)$$

with the relative Hilbert scheme of n points on the fibres of $\mathcal{C} \rightarrow H_\beta$.

It is natural to ask whether the localized perfect obstruction theory on $\text{Hilb}^n(\mathcal{C}/H_\beta)$ can be described directly without reference to the 3-fold X . This is investigated in the appendix of [KT1] by D. Panov, R. P. Thomas, and the author. Roughly speaking, one can embed $\text{Hilb}^n(\mathcal{C}/H_\beta)$ in an explicit compact smooth ambient space \mathcal{A} and realize it as the zero locus of a tautological section of a sheaf E on \mathcal{A} . If we assume

$$(1) \quad h^2(L) = 0 \text{ for all effective line bundles } L \text{ with } c_1(L) = \beta,$$

then E is a vector bundle on a Zariski open neighbourhood of $\text{Hilb}^n(\mathcal{C}/H_\beta)$. This induces a perfect obstruction theory on $\text{Hilb}^n(\mathcal{C}/H_\beta)$. This theory is the same as the one coming from localizing the *reduced* stable pair theory of X . “Reduced” means a trivial piece $H^{0,2}(S) \otimes \mathcal{O}$ corresponding to deformations of S outside the Noether-Lefschetz locus of β has been removed from the obstruction sheaf [KT1]. The invariants are denoted by $\mathcal{P}_{1-h+n, \beta}^{\text{red}}(S, [pt]^m) \in \mathbb{Z}(t)$, where $[pt] \in H_0(S)$ is the class of a point and t is the equivariant parameter coming from localization on X . We denote the corresponding generating function by

$$PT_\beta(S, [pt]^m)^{\text{red}} := \sum_{\chi} \mathcal{P}_\chi^{\text{red}}(S, [pt]^m) q^\chi.$$

When the geometric genus $p_g(S) = 0$, non-reduced and reduced invariants coincide and we omit the superscript *red*. If we assume

$$(2) \quad h^2(L) = 0 \text{ for all line bundles } L \text{ with } c_1(L) = \beta,$$

then E is a vector bundle on the whole of \mathcal{A} . This allows one to express the invariants as intersection numbers of tautological bundles on \mathcal{A} . These can be

computed in the following sense. Via wedging together and integrating over S , the classes $\beta, k \in H^2(S, \mathbb{Z})$, and $1 \in H^4(S, \mathbb{Z})$ give elements

$$[\beta], [k] \in \Lambda^2 H^1(S, \mathbb{Z})^*, \text{ and } [1] \in \Lambda^4 H^1(S, \mathbb{Z})^*.$$

Wedging together any combination produces an element

$$\Lambda^a[\beta] \wedge \Lambda^b[k] \wedge \Lambda^c[1] \in \Lambda^{2q(S)} H^1(S, \mathbb{Z})^* \cong \mathbb{Z}, \text{ where } a + b + 2c = q(S).$$

Here $q(S) = h^{0,1}(S)$ is the irregularity of the surface and the canonical isomorphism comes from choosing any integral basis of $H^1(S, \mathbb{Z})/\text{torsion} \subset H^1(S, \mathbb{R})$ compatible with the orientation coming from the complex structure.

Theorem 1.1. [KT2, Thm. 1.2] *Fixing q, p_g, m, n , there exists a universal function $F_{q,p_g,m,n}(\mathbf{x})$ with variables $\mathbf{x} := (x_1, x_2, x_3, x_4, \{x_{abc}\}_{a+b+2c=q}, t)$ such that for any S with $q(S) = q$, $p_g(S) = p_g$, and $\beta \in H^2(S, \mathbb{Z})$ satisfying Condition (2), $\mathcal{P}_{1-h+n,\beta}^{\text{red}}(S, [pt]^m)$ is equal to*

$$F_{q,p_g,m,n}(\beta^2, \beta.k, k^2, c_2(S), \{\Lambda^a[\beta] \wedge \Lambda^b[k] \wedge \Lambda^c[1]\}_{a+b+2c=q}, t).$$

Since the reduced stable pair invariants are still defined for S, β satisfying Condition (1), it is natural to ask whether Theorem 1.1 still holds in this case. We show by examples that this is *not* the case (Section 3).

Suppose $p_g(S) = 0$. If β or $k - \beta$ satisfies Condition (2), then one of $\mathcal{P}_{1-h+n,\beta}(S, [pt]^m)$, $\mathcal{P}_{1-h+n,k-\beta}(S, [pt]^m)$ is given by the universal formula of Theorem 1.1 and the other is zero. Therefore, the interesting case is when neither β nor $k - \beta$ satisfies Condition (2). Although stable pair invariants in classes $\beta, k - \beta$ generally cannot be expected to be universal individually, their difference turns out to be universal again (Section 4).

Theorem 1.2. *Theorem 1.1 is not true when we replace Condition (2) by (1). Next, fix S, β such that $p_g(S) = 0$ and neither β nor $k - \beta$ satisfies Condition (2). If $\beta(\beta - k) < 0$, then*

$$\mathcal{P}_{1-h+n,\beta}(S, [pt]^m) = \mathcal{P}_{1-h+n,k-\beta}(S, [pt]^m) = 0.$$

If $\beta(\beta - k) \geq 0$, then $\beta(\beta - k) = 0$, $q(S) = 1$, and

$$\mathcal{P}_{1-h+n,\beta}(S, [pt]^m) = \mathcal{P}_{1-h+n,k-\beta}(S, [pt]^m) = 0, \text{ for } m > 0$$

$$PT_\beta(S, [pt]^0) = PT_{k-\beta}(S, [pt]^0)(q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{2k(2\beta-k)} + \frac{1}{2}[2\beta - k](q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{2\beta^2}.$$

Here $[2\beta - k] \in \Lambda^2 H^1(S, \mathbb{Z})^* \cong \mathbb{Z}$ was defined above. The formula in Theorem 1.2 is invariant under $\beta \leftrightarrow k - \beta$. Examples of S, β with $p_g(S) = 0$, $\beta(\beta - k) \geq 0$, and neither β nor $k - \beta$ satisfying Condition (2) are given in Remark 4.5. Such surfaces are necessarily elliptic fibrations or blow-ups thereof.

We also study the *non-reduced* stable pair invariants of any S, β . These invariants are defined without any conditions on S, β and we denote them by $\mathcal{P}_{1-h+n, \beta}(S, [pt]^m) \in \mathbb{Z}(t)$. Although zero whenever S can be deformed to a surface where β is no longer of type $(1, 1)$, these invariants are interesting for classes such as $\beta = k$. In Section 4, we show:

Proposition 1.3. *For any S, β , and $m = \frac{\beta(\beta-k)}{2}$*

$$PT_\beta(S, [pt]^m) = t^m SW(\beta)(q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{2h-2},$$

where t is the equivariant parameter, $2h - 2 = \beta(\beta + k)$, and $SW(\beta)$ is the Seiberg-Witten invariant of S, β . In particular, the BPS spectrum¹ of S, β is

$$n_{h, \beta} = t^m SW(\beta), \quad n_{g, \beta} = 0 \text{ for } g \neq h.$$

Combining this proposition with the wall-crossing formula of M. Dür, A. Kabanov, and C. Okonek [DKO] gives the “duality formula” of Theorem 1.2. The occurrence of the Seiberg-Witten invariants is due to the Poincaré/SW correspondence conjectured by [DKO] and recently proved by H.-l. Chang and Y.-H. Kiem [CK]. Another application of this proposition is to Taubes’ GW/SW correspondence [Tau1, Tau2]. Let $m = \frac{\beta(\beta-k)}{2}$ as before. For any g , let $\overline{M}'_{g, m}(S, \beta)$ be the moduli space of stable maps with possibly disconnected domain curve and no collapsed connected components. This moduli space has a virtual normal bundle N^{vir} coming from localization on $X = K_S$ [GP]. Consider the following Gromov-Witten invariants of X

$$\begin{aligned} \mathcal{R}_{g, \beta}(X, [pt]^m) &= \int_{[\overline{M}'_{g, m}(S, \beta)]^{vir}} \frac{1}{e(N^{vir})} \prod_{i=1}^m \text{ev}_i^*[pt] \in \mathbb{Q}(t), \\ GW_\beta(X, [pt]^m) &= \sum_{g \geq h} \mathcal{R}_{g, \beta}(X, [pt]^m) u^{2g-2}. \end{aligned}$$

Proposition 1.4. *Fix any S, β with β irreducible. The GW/PT correspondence² for $GW_\beta(X, [pt]^m)$, $PT_\beta(X, [pt]^m)$ is equivalent the following equality*

$$GW_\beta(X, [pt]^m) = t^m SW(\beta)(2 \sin(u/2))^{2h-2}.$$

In particular, setting $-q = e^{iu}$, the leading coefficients of $GW_\beta(X, [pt]^m)$, $PT_\beta(X, [pt]^m)$ coincide if and only if

$$SW(\beta) = \int_{[\overline{M}'_{h, m}(S, \beta)]^{vir}} \prod_{i=1}^m \text{ev}_i^*[pt].$$

¹Defined in [PT1, Sect. 3.4], here applied to $PT_\beta(S, [pt]^m)$.

²I.e. [PT1, Conj. 3.3] but for X a non-compact Calabi-Yau 3-fold. The invariants are then defined by localization as above so we can put in point insertions. See also [MPT, Sect. 1.4].

We have a similar result for any S, β with $-K_S$ nef and β sufficiently ample (Remark 4.1). The results of this paper rely heavily on [DKO]. We take the opportunity to survey part of their work along the way.

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2. VIRTUAL CYCLES AND INVARIANTS

In this section, we recall the construction of the reduced perfect obstruction theory on $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$ when Condition (1) is satisfied [KT1, Appendix]. When Condition (2) is satisfied [KT1, Thm. A.7] gives a formula for the reduced virtual cycle. We use a result from [DKO] to show the formula still holds when Condition (1) is satisfied. The formula is not used in the proofs of Theorem 1.2 and Propositions 1.3, 1.4, but is of independent interest. We also give a formula for the non-reduced virtual cycle and recall the definition of the various invariants.

2.1. Reduced virtual cycle. We start with the natural embedding

$$\mathrm{Hilb}^n(\mathcal{C}/H_\beta) \hookrightarrow S^{[n]} \times H_\beta,$$

where $S^{[n]}$ is the Hilbert scheme of n points on S . A point (Z, C) lies in $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$ if and only if

$$s_C|_Z = 0 \in H^0(\mathcal{O}_Z(C)).$$

The family version of this is as follows. Let $\mathcal{Z} \subset S^{[n]} \times S$ be the universal subscheme and $\pi : S^{[n]} \times S \times H_\beta \rightarrow S^{[n]} \times H_\beta$ projection. Then $\pi_*(\mathcal{O}(\mathcal{C})|_{\mathcal{Z} \times H_\beta})$ is a vector bundle and it has a tautological section with zero locus $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$. This provides $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$ with a relative perfect obstruction theory over H_β . This construction does not immediately define an absolute perfect obstruction theory on $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$, because H_β can be singular.

Next, we embed H_β in a compact smooth ambient space. Let A be a sufficiently ample divisor and define $\gamma := [A] + \beta$. Then the Abel-Jacobi makes $H_\gamma := \mathrm{Hilb}_\gamma(S)$ into a projective bundle over the Picard variety $\mathrm{Pic}^\gamma(S)$. In particular, H_γ is smooth. Consider the closed embedding

$$H_\beta \hookrightarrow H_\gamma, \quad C \mapsto A \cup C.$$

A point D lies in the image of this map if and only if it contains A , i.e.

$$s_D|_A = 0 \in H^0(\mathcal{O}_A(D)).$$

The family version of this is as follows. Let $\mathcal{D} \rightarrow H_\gamma$ be the universal curve and $\pi : H_\gamma \times S \rightarrow H_\gamma$ projection. Then the *sheaf* $F := \pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$ has a tautological section with zero locus H_β . Assume Condition (1) is satisfied. Then $h^1(\mathcal{O}_A(A+C)) = 0$ for any $C \in H_\beta$. By semicontinuity and base change, $R^1\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$ is zero on a Zariski open neighbourhood of H_β . Hence F is a vector bundle on a Zariski open neighbourhood of H_β . This gives a perfect obstruction theory on H_β .

Both perfect obstruction theories can be combined to give a perfect obstruction theory on $\text{Hilb}^n(\mathcal{C}/H_\beta)$ [KT1, Thm. A.7]. The corresponding virtual cycle is denoted by $[\text{Hilb}^n(\mathcal{C}/H_\beta)]^{red}$ and referred to as the reduced virtual cycle.

If Condition (2) is satisfied, then $R^1\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A}) = 0$ on H_γ so F is a vector bundle on H_γ . Denoting the embedding $\text{Hilb}^n(\mathcal{C}/H_\beta) \hookrightarrow S^{[n]} \times H_\gamma$ by ι , one can show [KT1, Thm. A.7]

$$\iota_*[\text{Hilb}^n(\mathcal{C}/H_\beta)]^{red} = c_r(F) \cdot c_n(\pi_*(\mathcal{O}(\mathcal{D} - A)|_{\mathcal{Z} \times H_\gamma})),$$

where $r := \chi(\beta(A)) - \chi(\beta)$. Here $\chi(\beta)$ is the holomorphic Euler characteristic of any $\mathcal{O}(C)$, $C \in H_\beta$

$$2\chi(\beta) = \beta^2 - k\beta + 2\chi(\mathcal{O}_S)$$

and $\chi(\beta(A))$ is defined similarly. The virtual dimension is

$$v = \frac{\beta(\beta - k)}{2} + p_g(S) + n.$$

The following somewhat surprising observation follows from [DKO, Lem. 2.17].

Proposition 2.1. *Fix S, β such that Condition (1) is satisfied, $H_\beta \neq \emptyset$, and $\beta(\beta - k) \geq 0$. Then F is a vector bundle on H_γ even though $R^1\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$ is in general non-zero. Consequently*

$$\iota_*[\text{Hilb}^n(\mathcal{C}/H_\beta)]^{red} = c_r(F) \cdot c_n(\pi_*(\mathcal{O}(\mathcal{D} - A)|_{\mathcal{Z} \times H_\gamma}))$$

and its virtual dimension is $v = \frac{\beta(\beta - k)}{2} + p_g(S) + n$.

Proof. Let $p : \text{Pic}^\beta(S) \times S \rightarrow \text{Pic}^\beta(S)$ be projection and let \mathcal{P} be a choice of Poincaré bundle on $\text{Pic}^\beta(S) \times S$. Let

$$\mathbb{E} := [E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2]$$

be a resolution of $Rp_*\mathcal{P}$ by locally free sheaves. Then [DKO, Lem. 2.17] states $\ker d^1$ is locally free (Claim). We reproduce [DKO]'s proof of Claim below, but first finish the rest of the proof.

We take a resolution \mathbb{E} of $Rp_*\mathcal{P}$ of the following form. Let $[E^1 \xrightarrow{d^1} E^2]$ be a resolution of $Rp_*\mathcal{P}_A(A)$ by locally free sheaves and set $E^0 := p_*\mathcal{P}(A)$. Note

that $p_*\mathcal{P}(A)$ is a locally free by ampleness of A . We define \mathbb{E} by the following diagram of exact triangles

$$\begin{array}{ccccc} \mathbb{E} & \dashrightarrow & E^0 & \dashrightarrow & [E^1 \xrightarrow{d^1} E^2] \\ \downarrow & & \parallel & & \downarrow \cong \\ Rp_*\mathcal{P} & \longrightarrow & p_*\mathcal{P}(A) & \longrightarrow & Rp_*\mathcal{P}_A(A). \end{array}$$

Here \mathcal{P}_A is short hand for $\mathcal{P}|_{H_\gamma \times A}$. By Claim, $p_*\mathcal{P}_A(A) \cong \ker d^1$ is a locally free. Next, let \mathcal{Q} be a choice of Poincaré bundle on $\text{Pic}^\gamma(S)$. The Abel-Jacobi map

$$AJ : H_\gamma = \mathbb{P}(p_*\mathcal{Q}) \rightarrow \text{Pic}^\gamma(S)$$

is a projective bundle with tautological bundle $\mathcal{O}(1)$. Note that $\mathcal{Q}(1) \cong \mathcal{O}(\mathcal{D})$ on $H_\gamma \times S$, therefore

$$F = \pi_*\mathcal{O}(\mathcal{D}|_{H_\gamma \times A}) \cong AJ^*(p_*\mathcal{Q}_A)(1).$$

Since $\text{Pic}^\beta(S) \cong \text{Pic}^\gamma(S)$ sends $p_*\mathcal{P}_A(A)$ to $p_*\mathcal{Q}_A$ the proposition follows.

Next, we present [DKO]'s proof of Claim [DKO, Lem. 2.17]. If $H_{k-\beta} = \emptyset$, then $R^2p_*\mathcal{P} = 0$ so d^1 is surjective and $\ker d^1$ is locally free. Suppose $H_\beta, H_{k-\beta}$ are both non-empty. Condition (1) is equivalent to the statement that the images (Brill-Noether loci) of the two maps $H_{k-\beta} \rightarrow \text{Pic}^{k-\beta}(S) \cong \text{Pic}^\beta(S)$, $H_\beta \rightarrow \text{Pic}^\beta(S)$ are disjoint. In other words, their complements U, V satisfy $\text{Pic}^\beta(S) = U \cup V$. Moreover, for any $L \in \text{Pic}^\beta(S)$, we have $h^2(L) = 0$ when $L \in U$ and $h^0(L) = 0$ when $L \in V$. In other words

$$(3) \quad R^2p_*\mathcal{P}|_U = 0, \quad R^0p_*\mathcal{P}|_V = 0.$$

What about $R^1p_*\mathcal{P}$? To prove Claim, it suffices to show $R^1p_*\mathcal{P}|_V = 0$. Indeed, then $\ker d^1|_U$ is locally free ($d^1|_U$ is surjective) and $\ker d^1|_V = \text{im } d^0|_V$ is locally free ($(d^0)^*|_V$ is surjective).

Since H_β and $H_{k-\beta}$ are both nonempty, S cannot be rational because otherwise we get a section of K_S . Moreover, S cannot be ruled: for F the class of a fibre either $\beta.[F] < 0$ in which case $H_\beta = \emptyset$ or $\beta.[F] \geq 0$ in which case $(k-\beta).[F] < 0$ so $H_{k-\beta} = \emptyset$. Similarly, S cannot be the blow-up of a ruled surface. We conclude that the Kodaira dimension of S is ≥ 0 . Therefore $\chi(\mathcal{O}_S) \geq 0$ and

$$\text{rk } Rp_*\mathcal{P} = \frac{\beta(\beta-k)}{2} + \chi(\mathcal{O}_S) \geq 0.$$

On the other hand, (3) implies $\text{rk } Rp_*\mathcal{P} \leq 0$ so $\text{rk } Rp_*\mathcal{P} = 0$. This means $R^1p_*\mathcal{P}$ is torsion. Since $R^1p_*\mathcal{P}|_V$ is a subsheaf of the locally free sheaf $E^1/\text{im } d^0|_V$ ($(d^0)^*|_V$ is surjective), it is zero. \square

2.2. Non-reduced virtual cycle. The Hilbert scheme H_β carries another natural virtual cycle coming from a perfect obstruction theory of the form

$$(R\pi_*\mathcal{O}_{\mathcal{C}}(\mathcal{C}))^\vee \rightarrow \mathbb{L}_{H_\beta}.$$

This perfect obstruction theory is studied in [DKO] and also appears in [KT1, Appendix]. Combining this perfect obstruction theory with the relative perfect obstruction theory on $S^{[n]} \times H_\beta$ induced by $\pi_*(\mathcal{O}(\mathcal{C})|_{\mathcal{Z} \times H_\beta})$ gives another perfect obstruction theory on $\text{Hilb}^n(\mathcal{C}/H_\beta)$. See diagram (84) of [KT1, Appendix] for details. These perfect obstruction theories are defined for *any* S, β . We refer to the corresponding virtual cycles $[H_\beta]^{vir}$, $[\text{Hilb}^n(\mathcal{C}/H_\beta)]^{vir}$ as non-reduced virtual cycles. They coincide with the reduced virtual cycles $[H_\beta]^{red}$, $[\text{Hilb}^n(\mathcal{C}/H_\beta)]^{red}$ when $p_g(S) = 0$. It is shown in [KT1, Appendix], that $[\text{Hilb}^n(\mathcal{C}/H_\beta)]^{vir}$ coincides with the cycle induced by localizing non-reduced stable pair theory on $X = K_S$, whereas $[\text{Hilb}^n(\mathcal{C}/H_\beta)]^{red}$ coincides with the cycle induced by localizing reduced stable pair theory on $X = K_S$.

Let $\iota : \text{Hilb}^n(\mathcal{C}/H_\beta) \hookrightarrow S^{[n]} \times H_\beta$ be the embedding of the previous section. Even though H_β can be singular, we have the following:

Proposition 2.2. *For any S, β*

$$\iota_*[\text{Hilb}^n(\mathcal{C}/H_\beta)]^{vir} = (S^{[n]} \times [H_\beta]^{vir}).c_n(\pi_*\mathcal{O}((\mathcal{C})|_{\mathcal{Z} \times H_\beta}))$$

and its virtual dimension is $v = \frac{\beta(\beta-k)}{2} + n$.

For the proof, we need the following lemma.

Lemma 2.3. *Let $\pi : M \rightarrow B$ be a flat morphism of \mathbb{C} -schemes of finite type with B projective. Let $E^\bullet \rightarrow \mathbb{L}_B$, $F^\bullet \rightarrow \mathbb{L}_M$ be perfect obstruction theories. Suppose there exists a smooth projective variety A and a rank r vector bundle V on $A \times B$ with regular³ section s such that $M = s^{-1}(0) \subset A \times B$ and $\pi : M \rightarrow B$ commutes with projection $\pi_B : A \times B \rightarrow B$. This induces a canonical relative perfect obstruction theory $G^\bullet \rightarrow \mathbb{L}_{M/B}$ of the form $G^\bullet = \{V^*|_M \rightarrow \pi_A^*(\Omega_A)|_M\}$. Suppose there exists an exact triangle*

$$\pi^*E^\bullet \longrightarrow F^\bullet \longrightarrow G^\bullet.$$

Denote inclusion by $\iota : M \hookrightarrow A \times B$. Then

$$\iota_*[M]^{vir} = (A \times [B]^{vir}).c_r(V).$$

Proof. For any perfect obstruction theory $F^\bullet \rightarrow \mathbb{L}_M$ with M projective, the following formula holds [Sie, Thm. 4.6] (see also [Pid])

$$(4) \quad [M]^{vir} = \left\{ s_\bullet(F^{\bullet\vee})_{c_F(M)} \right\}_v.$$

³As defined in [Ful, B.3.4].

Here $s_\bullet(\cdot)$ is the total Segre class, v is the virtual dimension of M , and $c_F(M)$ is Fulton's canonical class which is defined as follows. Take any embedding $M \subset \mathcal{A}$ into a smooth variety \mathcal{A} , then

$$c_F(M) := c_\bullet(T_{\mathcal{A}}|_M)s_\bullet(C_{M/\mathcal{A}}),$$

where $C_{M/\mathcal{A}}$ is the normal cone of $M \subset \mathcal{A}$. This definition is independent of choice of embedding [Ful, Ex. 4.2.6]. Take an embedding $B \subset C$ into a smooth variety and consider

$$M \subset A \times B \subset A \times C =: \mathcal{A}.$$

By the assumptions

$$s_\bullet(F^{\bullet\vee}) = \pi^*(s_\bullet(E^{\bullet\vee})) \frac{c_\bullet(V|_M)}{\pi_A^*(c_\bullet(T_A))|_M}.$$

Since $M \subset A \times B$ is cut out by a regular section of V , we have $C_{M/A \times B} \cong N_{M/A \times B} \cong V|_M$. Consider the following short exact sequence of cones

$$N_{M/A \times B} \longrightarrow C_{M/A \times C} \longrightarrow C_{A \times B/A \times C}|_M.$$

We deduce

$$c_F(M) = \pi_A^*(c_\bullet(T_A))|_M \pi^*(c_\bullet(T_C|_B)) \frac{\pi^*s_\bullet(C_{B/C})}{c_\bullet(V|_M)}.$$

Formula (4) implies

$$[M]^{vir} = \left\{ \pi^*(s_\bullet(E^{\bullet\vee})c_\bullet(T_C|_B)s_\bullet(C_{B/C})) \right\}_v = \pi^*[B]^{vir}.$$

The second equality follows by applying (4) to $E^\bullet \rightarrow \mathbb{L}_B$. The projection formula gives

$$\iota_*[M]^{vir} = (A \times [B]^{vir}).\iota_*[M].$$

Since $M \subset A \times B$ is cut out by a regular section of V , we have $\iota_*[M] = c_r(V)$ [Ful, Prop. 14.1] and the proposition is proved. \square

Proof of Proposition 2.2. Diagram (86) of [KT1, Appendix] provides the required exact triangle. It is left to show $\text{Hilb}^n(\mathcal{C}/H_\beta) \rightarrow H_\beta$ is flat and the tautological section σ of $\pi_*(\mathcal{O}(\mathcal{C})|_{\mathcal{Z} \times H_\beta})$ is regular. The fibre of the morphism $\text{Hilb}^n(\mathcal{C}/H_\beta) \rightarrow H_\beta$ over $C \in H_\beta$ is $C^{[n]}$. The scheme $C^{[n]}$ is cut out by the tautological section of the tautological bundle $L^{[n]}$ where $L := \mathcal{O}(C)$. Moreover, $C^{[n]} \subset S^{[n]}$ always has codimension n (see [KT1, Footnote 18], which uses [AIK, Iar]). Therefore $\sigma|_{S^{[n]} \times \{C\}}$ is regular for all $C \in H_\beta$. From this one can deduce $\text{Hilb}^n(\mathcal{C}/H_\beta) \rightarrow H_\beta$ is flat and σ is regular. \square

2.3. Invariants. The reduced virtual cycle $[P_\chi(S, \beta)]^{red} = [\text{Hilb}^n(\mathcal{C}/H_\beta)]^{red}$ can be used to define invariants as follows. Let $\mathcal{O} \rightarrow \mathbb{F}$ be the universal stable pair on $S \times P_\chi(S, \beta)$. For each $\sigma \in H^*(S, \mathbb{Z})$, define the 0th descendent insertion $\tau_0(\sigma)$ by

$$\tau_0(\sigma) := \pi_{P*}(\pi_S^*(\sigma) \cdot c_1(\mathbb{F})) \in H^*(P_\chi(S, \beta), \mathbb{Z}).$$

Another important class on $P_\chi(S, \beta)$ comes from the virtual normal bundle. Localizing reduced stable pair theory on $X = K_S$, one obtains a virtual normal bundle N^{vir} on $P_\chi(X, \beta)^{\mathbb{C}^*}$ [GP]. Its top Chern class $e(N^{vir})$ is an element of

$$H_{\mathbb{C}^*}^*(P_\chi(X, \beta), \mathbb{Q}) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, t^{-1}] \cong H^*(P_\chi(X, \beta)^{\mathbb{C}^*}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[t, t^{-1}].$$

An explicit description of $e(N^{vir})$ restricted to $P_\chi(S, \beta)$ can be found in [KT2, Sect. 2.1]. For each $\sigma_1, \dots, \sigma_m \in H^*(S, \mathbb{Z})$, we define reduced stable pair invariants of S

$$\mathcal{P}_{\chi, \beta}^{red}(S, \sigma_1 \dots \sigma_m) := \int_{[P_\chi(S, \beta)]^{red}} \frac{1}{e(N^{vir})} \prod_{i=1}^m \tau_0(\sigma_i) \in \mathbb{Z}(t).$$

Replacing $[P_\chi(S, \beta)]^{red}$ by $[P_\chi(S, \beta)]^{vir}$, we can similarly define non-reduced invariants $\mathcal{P}_{\chi, \beta}(S, \sigma_1 \dots \sigma_m)$. The virtual normal bundle obtained from localizing reduced or non-reduced stable pair theory on X are the same [KT1, Prop. 3.4].

In the case $n = 0$, these non-reduced invariants were studied in [DKO]. We recall their definitions. Consider the two Abel-Jacobi maps

$$\begin{aligned} AJ^+ : H_\beta &\rightarrow \text{Pic}^\beta(S), \\ AJ^- : H_{k-\beta} &\rightarrow \text{Pic}^{k-\beta}(S) \cong \text{Pic}^\beta(S). \end{aligned}$$

Then one can define Poincaré invariants [DKO, Def. 3.1]

$$\begin{aligned} P_S^+(\beta) &:= AJ_*^+ \left(\sum_i c_1(\mathcal{O}(\mathcal{C})|_{H_\beta \times \{pt\}})^i \cap [H_\beta]^{vir} \right), \\ P_S^-(\beta) &:= (-1)^{\chi(\mathcal{O}_S) + \frac{\beta(\beta-k)}{2}} AJ_*^- \left(\sum_i (-1)^i c_1(\mathcal{O}(\mathcal{C})|_{H_{k-\beta} \times \{pt\}})^i \cap [H_{k-\beta}]^{vir} \right). \end{aligned}$$

In the first line, \mathcal{C} denotes the universal divisor over H_β and in the second line, the universal divisor over $H_{k-\beta}$. Note that $P_S^\pm(\beta) \in \Lambda^* H^1(S, \mathbb{Z})^*$. The authors conjectured $P_S^\pm(\beta)$ are equal to Seiberg-Witten invariants [DKO, Conj. 5.3]. Using a wall-crossing formula and blow-up formula for $P_S^\pm(\beta)$, they prove the conjecture in several cases. The proof of the conjecture was recently completed by [CK] using cosection localization. We denote the degree $2q(S)$ part of $P_S^+(\beta) \in \Lambda^* H^1(S, \mathbb{Z})^*$ by $SW(\beta) \in \mathbb{Z}$. This is the original Seiberg-Witten invariant of S, β (see [Wit, Moo]).

Remark 2.4. If S, β satisfies Condition (2), then $c_r(F) = c_r(R\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A}))$. This is the basis for the proof of Theorem 1.1 [KT2, Thm. 1.2]. If S, β satisfies Condition (1), then [KT2, Sect. 4] and Prop. 2.1 gives the following weak universality result. Suppose $\beta(\beta - k) \geq 0$ and $H_\beta \neq \emptyset$. Let \mathcal{P} be a choice of normalized Poincaré bundle on $\text{Pic}^\beta(S)$, i.e. $\mathcal{P}|_{\text{Pic}^\beta(S) \times \{pt\}} \cong \mathcal{O}$. Let $p : \text{Pic}^\beta(S) \times S \rightarrow \text{Pic}^\beta(S)$ be projection and let $\tau_{\leq 1}$ be the truncation functor. Then the reduced stable pair invariants $\mathcal{P}_{1-h+n, \beta}^{\text{red}}(S, [pt]^m)$ are given by universal functions in the variables of Theorem 1.1 and integrals of the form

$$\int_{S \times \text{Pic}^\beta(S)} M(c_1(\mathcal{P}), c_1(S), c_2(S), c_\bullet(p! \mathcal{P}), p^* p_*(c_1(\mathcal{P})^j c_k(S)^l), c_\bullet(\tau_{\leq 1} p! \mathcal{P})),$$

where M is any monomial in the indicated variables. In general, we cannot evaluate these integrals any further because of the appearance of $\tau_{\leq 1}$. Here the obvious pull-backs from the factors of $S \times \text{Pic}^\beta(S)$ are suppressed. \circlearrowright

3. ELLIPTIC FIBRATIONS

In this section, we give several counter-examples to universality when only Condition (1) is satisfied. This proves the first statement of Theorem 1.2. All examples are elliptic fibrations.

Let $\pi : S \rightarrow C$ be an elliptic fibration over a curve of genus g [BPV, Ch. V.6]. We are only interested in the case S, C are algebraic. The generic fibre F is a smooth elliptic curve and we denote by $m_1 F_1, \dots, m_r F_r$ the multiple fibres. The canonical divisor is given by

$$(5) \quad K_S = \pi^* D + \sum_{i=1}^r (m_i - 1) F_i,$$

where D is a divisor of degree $2g - 2 + \chi(\mathcal{O}_S)$ on C [BPV, Cor. 12.3]. In this section, we will make frequent use of logarithmic transformations [BPV, Ch. V.13]. Given a generic⁴ point $x \in C$, a logarithmic transformation replaces the fibre F over x by a multiple mF , $m > 1$. The new elliptic fibration $\pi' : S' \rightarrow C$ has fibre mF over $x \in C$ and the restrictions $\pi^{-1}(C \setminus \{x\})$, $\pi'^{-1}(C \setminus \{x\})$ are biholomorphic as fibre bundles over $C \setminus \{x\}$. One should not think of a logarithmic transformation as a sort of birational transformation. The topology of S can change and S can even become non-algebraic [BPV, Ch. V.13].

Example 3.1. Let $\mathbb{P}^1 \subset |\mathcal{O}(3)|$ be a generic pencil of cubics on \mathbb{P}^2 and let $S \rightarrow \mathbb{P}^1$ be the universal curve. This is a rational elliptic fibration so $q(S) =$

⁴In fact, logarithmic transformations can be applied to any fibre of type I_b .

$p_g(S) = 0$ and $K_S = -F$ (equation (5)). We take $\beta = 6k$. Clearly $|6K_S| = \emptyset$ so $SW(\beta) = 0$. Let S' be obtained from S by replacing one general fibre F by a double fibre $2F_1$ and another by a triple fibre $3F_2$. Then S' is one of the famous Dolgachev surfaces⁵. The surface S' is known to be algebraic satisfying $q(S') = p_g(S') = 0$ and $K_{S'} = -F + F_1 + 2F_2$ (equation (5)). In the Chow group, one has relations $2F_1 = 3F_2 = F$ so $k' = \frac{1}{6}[F]$ in $H^2(S', \mathbb{Q})$. Taking $\beta' = 6k'$, we see that $|6K_{S'}| = |F| \neq \emptyset$, whereas $|K_{S'} - 6K_{S'}| = |-5K_{S'}| = \emptyset$. Consequently, $SW(k' - \beta') = 0$. The wall-crossing formula⁶ [DKO, Prop. 4.1] states $SW(\beta') - SW(k' - \beta') = 1$, so $SW(\beta') = 1$. Since the Chern numbers of S, β and S', β' are the same, this is a counter-example to universality. Note that this does not contradict Theorem 1.1. Indeed

$$h^2(\mathcal{O}(6K_S)) = h^0(\mathcal{O}(-5K_S)) = h^0(\mathcal{O}(5F)) \neq 0,$$

so β' satisfies Condition (2) but β only satisfies Condition (1). \oslash

In order to find more counter-examples to universality, we use the following result due to R. Friedman and J. W. Morgan [FM, Prop. 4.4], and [DKO, Prop. 4.8]).

Proposition 3.2 (Friedman-Morgan, Dürr-Kabanov-Okonek). *Suppose $\beta \in H^2(S, \mathbb{Z})$ satisfies $\beta^2 = \beta \cdot [F] = 0$. Then*

$$SW(\beta) = \sum_{\substack{d[F] + \sum_i a_i [F_i] = \beta \\ d \geq 0, 0 \leq a_i < m_i}} (-1)^d \binom{2g - 2 + \chi(\mathcal{O}_S)}{d}.$$

Here we should recall [FM]'s conventions on binomial coefficients. For each $b \geq 0$, define $\binom{a}{b} = \frac{1}{b!} a(a-1) \cdots (a-b+1)$. In particular, $\binom{a}{b} = 1$ for $b = 0$, $\binom{a}{b} = 0$ for $0 \leq a < b$, and $\binom{-a}{b} = (-1)^b \binom{a+b-1}{b}$.

Example 3.3. Let S be an hyper-elliptic surface and $\beta = d[F]$ for any $d \in \mathbb{Z}_{\geq 0}$. Note that $q(S) = 1$ and $p_g(S) = 0$. Proposition 3.2 implies $SW(\beta) = 0$ for $d > 0$ and $SW(\beta) = 1$ for $d = 0$. Since $\beta^2 = \beta \cdot k = k^2 = c_2(S) = [\beta] = [k] = 0$ for any $d \geq 0$, this also provides a counter-example to universality. Although $\mathcal{O}(K_S)$ is a non-trivial torsion element of $A^1(S)$, its class $k = 0 \in H^2(S, \mathbb{Z})$. The class $\beta = 0$ satisfies Condition (1) but not Condition (2) since $h^2(\mathcal{O}(K_S)) \neq 0$. Hence, there is no contradiction with Theorem 1.1. \oslash

⁵The surfaces S, S' provide homeomorphic compact simply connected 4-manifolds. S. K. Donaldson famously proved their C^∞ -structures are different [Don]. One can also establish this by showing their Seiberg-Witten invariants are distinct (see [Moo]).

⁶Since we need it for the proof of Theorem 1.2, we review [DKO]'s wall-crossing formula and its proof for any S, β with $p_g(S) = 0$ in Section 4.2.

Finally, we apply Proposition 3.2 to a special class of logarithmic transformations discussed in [DKO, Sect. 4.2]. They will provide more interesting counter-examples to universality. We recall the construction. Fix an elliptic curve $F = \mathbb{C}/\Gamma$ with lattice $\Gamma = \langle 1, \omega \rangle \subset \mathbb{C}$. We apply logarithmic transformations to $\mathbb{P}^1 \times F \rightarrow \mathbb{P}^1$ as follows. Fix a point $t_1 \in \mathbb{P}^1$ and an m_1 -torsion point $\zeta_1 \in F$, $m_1 > 0$. The logarithmic transformation $L_{t_1}(m_1, \zeta_1)(\mathbb{P}^1 \times F)$ replaces the fibre over t_1 by $m_1 F_1$. Continuing in this fashion with other distinct points $t_2, \dots, t_r \in \mathbb{P}^1$, one obtains a smooth compact complex surface

$$S := L_{\underline{t}}(\underline{n}, \underline{\zeta})(\mathbb{P}^1 \times F),$$

which is an elliptic fibration over \mathbb{P}^1 . It has generic fibre F and multiple fibres $m_1 F_1, \dots, m_r F_r$. The following proposition [DKO, Sect. 4.2] summarizes the relevant geometry.

Proposition 3.4 (Dürr-Kabanov-Okonek). *Suppose ζ_1, \dots, ζ_r are of the form $\zeta_i = \frac{u_i + v_i \omega}{m_i}$ for integers u_i, v_i satisfying $\gcd(m_i, u_i, v_i) = 1$.*

- (1) *The surface S is projective if and only if $\sum_{i=1}^r \zeta_i = 0$.*
- (2) *Suppose (1) is satisfied. Then $H^2(S, \mathbb{Z}) \cong \mathbb{Z} \oplus G$, where G is the free abelian group generated by $[F], [F_1], \dots, [F_r]$ modulo the relations*

$$m_1[F_1] = \dots = m_r[F_r] = [F],$$

$$u_1[F_1] + \dots + u_r[F_r] = 0, \quad v_1[F_1] + \dots + v_r[F_r] = 0.$$

- (3) *Suppose (1) is satisfied. Let $\Gamma' \subset \mathbb{C}$ be the lattice generated by the elements $1, \omega, \zeta_1, \dots, \zeta_r$ and consider the Albanese map $\text{Alb} : S \rightarrow \text{Alb}(S)$. Then there exists an isomorphism $\text{Alb}(S) \cong \mathbb{C}/\Gamma'$ such that the following diagram commutes*

$$\begin{array}{ccc} F & \xrightarrow{\text{Alb}|_F} & \text{Alb}(S) \\ \parallel & & \downarrow \cong \\ \mathbb{C}/\Gamma & \longrightarrow & \mathbb{C}/\Gamma', \end{array}$$

where the bottom map is induced by $\Gamma \subset \Gamma'$.

Example 3.5. Take⁷ $\zeta_1 = \frac{1+\omega}{3}$, $\zeta_2 = \frac{1}{3}$, $\zeta_3 = \frac{1}{3}$, and $\zeta_4 = -\frac{3+\omega}{3}$. By Proposition 3.4, S is projective, $[F] = 3[F_1]$, $[F_4] = [F_1]$, $[F_3] = 2[F_1] - [F_2]$, and

$$\begin{aligned} H^2(S, \mathbb{Z}) &\cong \mathbb{Z} \oplus \langle [F_1], [F_2] \mid 3[F_1] = 3[F_2] \rangle_{\mathbb{Z}} \\ &\cong \mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/3\mathbb{Z}. \end{aligned}$$

⁷This example is used by [DKO] to provide a surface S with $p_g(S) = 0$ and $SW(k) \neq 0$.

By equation (5), $K_S = 2F_1$ and $k = \frac{2}{3}[F] \in H^2(S, \mathbb{Q})$. We fix $\beta = n[F_1] + \epsilon[F_2]$ with $n \in \mathbb{Z}_{\geq 0}$ and $\epsilon = 0, 1, 2$. The surface S satisfies $q(S) = 1$ and $p_g(S) = 0$. Clearly, $\beta^2 = \beta.k = k^2 = c_2(S) = 0$. Let E be the fibre of $S \rightarrow \text{Alb}(S)$. Then Proposition 3.4 (3) implies $E.F = 9$. Hence $[\beta] = \beta.E = 3(n + \epsilon)$. By Proposition 3.2

$$SW(\beta) = \sum_{\substack{(3d + a_1 + 2a_3 + a_4)[F_1] + (a_2 - a_3)[F_2] = n[F_1] + \epsilon[F_2] \\ d \geq 0, a_i = 0, 1, 2}} (d + 1).$$

For all $(n, \epsilon) \notin \{(0, 0), (1, 0), (2, 0), (0, 1)\}$, this is equal to

$$SW(\beta) = [\beta] - 3.$$

For $(n, \epsilon) = (0, 0), (1, 0), (2, 0), (0, 1)$, we get the sporadic values $SW(\beta) = 1, 2, 4, 1$. This gives another counter-example to universality. \oslash

4. PROOFS AND COMMENTS

4.1. Proof of Propositions 1.3, 1.4 and comments.

Proof of Proposition 1.3. Since the virtual dimension of $[H_\beta]^{vir}$ is $m = \frac{\beta(\beta-k)}{2}$, the cycle

$$c_1(\mathcal{O}(\mathcal{C})|_{H_\beta \times \{pt\}})^m \cap [H_\beta]^{vir}$$

is 0-dimensional. Therefore, it is of the form $\sqcup_i m_i \{C_i\}$ with $C_i \in H_\beta$. Define $L_i := \mathcal{O}(C_i)$. In [KT2, Eqn. (12)], we give an explicit class on $H_\beta \times S^{[n]}$, which restricts to $1/e(N^{vir})$ on $\text{Hilb}^n(\mathcal{C}/H_\beta)$. Restricting this expression to $S^{[n]} \times \{C_i\} \subset S^{[n]} \times H_\beta$ gives

$$(-1)^{n_t m} \frac{c_\bullet(T_{S^{[n]}})}{c_\bullet(L_i^{[n]}},$$

where t is the equivariant parameter. Together with Proposition 2.2, this implies

$$\mathcal{P}_{1-h+n, \beta}(S, [pt]^m) = (-1)^{n_t m} \sum_i m_i \int_{S^{[n]}} c_n(L_i^{[n]}) \frac{c_\bullet(T_{S^{[n]}})}{c_\bullet(L_i^{[n]}}.$$

By [EGL], each integral over $S^{[n]}$ is given by a universal polynomial in $\beta^2, \beta.k, k^2, c_2(S)$ and therefore independent of i . Since $SW(\beta) = \sum_i m_i$, we obtain

$$\mathcal{P}_{1-h+n, \beta}(S, [pt]^m) = (-1)^{n_t m} SW(\beta) \int_{S^{[n]}} c_n(L^{[n]}) \frac{c_\bullet(T_{S^{[n]}})}{c_\bullet(L^{[n]}},$$

where $L := L_i$ for any choice of i . By [EGL], for any S, L with $c_1(L) = \beta$, the integral is given by a universal polynomial P_n in $\beta^2, \beta.k, k^2, c_2(S)$. Moreover,

for any S, L with $c_1(L) = \beta$ and L *very ample*, the integral can be computed as follows. By very ampleness, we can write $L = \mathcal{O}(C)$ with $C \subset S$ irreducible and smooth. Then $C^{[n]} \subset S^{[n]}$ is cut out smoothly and transversally by a tautological section of $L^{[n]}$ and the integral is equal to $e(C^{[n]})$. These Euler characteristics are given by the well-known expression

$$\sum_{n=0}^{\infty} e(C^{[n]}) q^n = (1 - q)^{2h-2}.$$

By universality, the polynomials in $\beta^2, \beta.k$ obtained from this expression are equal to the P_n . \square

Proof of Proposition 1.4. Since β is irreducible, $P_\chi(X, \beta)^{\mathbb{C}^*} \cong P_\chi(S, \beta)$ for all χ . Hence $PT_\beta(X, [pt]^m) = PT_\beta(S, [pt]^m)$ and the result follows from Proposition 1.3. Note that the equivariant parameter t of the leading term of both generating functions match by [KT1, Lem. 3.3]. \square

Remark 4.1. The following is a variation on Proposition 1.4. Fix any S, β such that $-K_S$ is nef and β is sufficiently ample⁸. Assume the GW/PT correspondence⁹ holds for $GW_\beta(X, [pt]^m)$, $PT_\beta(X, [pt]^m)$ and the BPS spectrum of X is finite¹⁰. Then

$$GW_\beta(X, [pt]^m) = t^m SW(\beta) (2 \sin(u/2))^{2h-2},$$

$$SW(\beta) = \int_{[\overline{M}'_{h,m}(S, \beta)]^{vir}} \prod_{i=1}^m \text{ev}_i^*[pt].$$

The proof goes as follows. Since $h \geq 1$ and the BPS spectrum is assumed finite, applying the coordinate transformation $-q = e^{iu}$ to $GW_\beta(X, [pt]^m)$ gives a Laurent *polynomial* in q . Moreover, it is of the form

$$(6) \quad aq^{-b} + \cdots + aq^b,$$

for some $b \geq 0$. By [KT1, Prop. 5.1], we have $P_\chi(X, \beta)^{\mathbb{C}^*} \cong P_\chi(S, \beta)$ for all $\chi \leq h - 1$. Combining this with Proposition 1.3 and (6) gives the result. \oslash

Remark 4.2. One can speculate that for *any* algebraic S, β , Taubes' GW/SW correspondence follows from the GW/PT correspondence. This requires dealing with other components of $P_\chi(X, \beta)^{\mathbb{C}^*}$. Conversely, one can try to derive cases of the GW/PT correspondence for $X = K_S$ from Taubes' GW/SW correspondence as is done in Proposition 1.4. \oslash

⁸I.e. β such that $h \geq 1$ and $(4h - 3)$ -very ample [KT2, Prop. 5.1].

⁹The GW/PT correspondence has been proved in many cases [MOOP, MPT, PP1, PP2].

¹⁰I.e. after writing $GW_\beta(X, [pt]^m)$ in BPS form [GV1, GV2], [PT1, (3.13)], we assume there are only finitely many nonzero $n_{g, \beta'}$.

4.2. Wall-crossing formula for Poincaré invariants. We recall the wall-crossing formula for Poincaré invariants $P_S^\pm(\beta)$ [DKO, Thm. 3.16].

Theorem 4.3 (Dürr-Kabanov-Okonek). *Let S be a surface with $p_g(S) = 0$. Let \mathcal{P} be a choice of normalized Poincaré bundle on $\text{Pic}^\beta(S)$. Denote projection by $p : \text{Pic}^\beta(S) \times S \rightarrow \text{Pic}^\beta(S)$. Then*

$$P_S^+(\beta) - P_S^-(\beta) = \sum_{i \geq 1 - \chi(\beta)} s_i(p! \mathcal{P}).$$

Proof. We reproduce the proof of [DKO]. If $\frac{\beta(\beta-k)}{2} < 0$, then a Grothendieck-Riemann-Roch computation shows that RHS is zero ([DKO, Lem. 3.13] or [KT2, Sect. 4]) and the theorem is trivial. Assume $\frac{\beta(\beta-k)}{2} \geq 0$.

As in Section 2, we denote by \mathcal{P}, \mathcal{Q} choices of Poincaré bundle on $\text{Pic}^\beta(S), \text{Pic}^\gamma(S)$. Consider $H_\beta \subset H_\gamma = \mathbb{P}(p_* \mathcal{Q})$ and let $h = c_1(\mathcal{O}(1))$. If $H_\beta \neq \emptyset$, then Proposition 2.1 implies

$$AJ_*(c_1(\mathcal{O}(\mathcal{C})|_{H_\beta \times \{pt\}})^i \cap [H_\beta]^{vir}) = AJ_*(c_r(F)h^i) = s_{i-\chi(\beta)+1}(\tau_{\leq 1} p! \mathcal{P}),$$

with $\tau_{\leq 1}$ truncation. The second equality follows at once from [DKO, Prop. 2.18] or [KT2, Lem. 4.3]. A similar formula holds for $[H_{k-\beta}]^{vir}$ when $H_{k-\beta} \neq \emptyset$.

If β satisfies Condition (2), then $H_{k-\beta} = \emptyset$ and $s_i(\tau_{\leq 1} p! \mathcal{P}) = s_i(p! \mathcal{P})$. The formula follows. If $k - \beta$ satisfies Condition (2), then $H_\beta = \emptyset$ and the formula follows similarly using Serre duality $Rp_* \mathcal{P}^*(K_S) \cong (Rp_* \mathcal{P}[2])^\vee$. We are left with the case neither β nor $k - \beta$ satisfies Condition (2). Then proving the wall-crossing formula is equivalent to showing

$$\sum_{i \geq 1 - \chi(\beta)} \left(s_i(\tau_{\leq 1} p! \mathcal{P}) + (-1)^i s_i(\tau_{\leq 1} p! \mathcal{P}^*(K_S)) \right) = \sum_{i \geq 1 - \chi(\beta)} s_i(p! \mathcal{P}).$$

In the proof of Proposition 2.1, we saw that $H_\beta, H_{k-\beta}$ both non-empty implies $\chi(\mathcal{O}_S) = 0$ and $\beta(\beta - k) = 0$. Since $p_g(S) = 0$, we have $q(S) = 1$. Since $s_1(\tau_{\leq 1} p! \mathcal{P}) = c_1(R^1 p_* \mathcal{P}) - c_1(p_* \mathcal{P})$, it suffices to show $s_1(\tau_{\leq 1} p! \mathcal{P}^*(K_S)) = c_1(R^2 p_* \mathcal{P})$.

Take a locally free resolution $[E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2]$ of $Rp_* \mathcal{P}$. Then Serre duality $Rp_* \mathcal{P}^*(K_S) \cong (Rp_* \mathcal{P}[2])^\vee$ implies

$$\begin{aligned} s_1(\tau_{\leq 1} p! \mathcal{P}^*(K_S)) &= c_1(\ker(d^{0*})) - c_1(E^{2*}) = c_1(E^2) + c_1((\text{coker } d^0)^*), \\ c_1(R^2 p_* \mathcal{P}) &= c_1(E^2) - c_1(\text{im } d^1) = c_1(E^2) + c_1((E^1 / \ker d^1)^*). \end{aligned}$$

Since $R^1 p_* \mathcal{P}$ is torsion (proof of Prop. 2.1), dualizing the short exact sequence

$$0 \rightarrow R^1 p_* \mathcal{P} \rightarrow \text{coker } d^0 \rightarrow E^1 / \ker d^1 \rightarrow 0$$

shows $(\text{coker } d^0)^* \cong (E^1 / \ker d^1)^*$. \square

4.3. Proof of Theorem 1.2 and comments.

Proof of Theorem 1.2. The failure of universality is illustrated by Examples (3.1), (3.3), and (3.5). Fix S, β such that $p_g(S) = 0$ and neither β nor $k - \beta$ satisfies Condition (2). If $\beta(\beta - k) < 0$, the virtual dimensions of $[H_\beta]^{vir}$ and $[H_{k-\beta}]^{vir}$ are zero and we use Proposition 2.2. Assume $\beta(\beta - k) \geq 0$. We saw in the proof of Theorem 4.3 that this implies $q(S) = 1$ and $\beta(\beta - k) = 0$. By Proposition 2.2, the invariants are zero when point insertions are present ($m > 0$). In the case $m = 0$, Proposition 1.3 implies

$$PT_\beta(S, [pt]^0) = P_S^+(\beta)(q^{1/2} + q^{-1/2})^{2\beta^2},$$

$$PT_{k-\beta}(S, [pt]^0) = P_S^+(k - \beta)(q^{1/2} + q^{-1/2})^{2(k-\beta)^2} = P_S^-(\beta)(q^{1/2} + q^{-1/2})^{2(k-\beta)^2}.$$

The result follows from Theorem 4.3 and a Grothendieck-Riemann-Roch computation giving $s_1(p_! \mathcal{P}) = \frac{1}{2}[2\beta - k]$. \square

Remark 4.4. One can consider reduced stable pair invariants with other insertion classes such as $\mathcal{P}_{1-h+n, \beta}^{red}(S, [pt]^m[\gamma_1] \dots [\gamma_{2q(S)}])$, where $\gamma_1, \dots, \gamma_{2q(S)} \in H_1(S)/\text{torsion}$ is an integral oriented basis [KT2, Sect. 3]. The H_1 -insertions cut H_β down to a linear system $|L| \subset H_\beta$. Fix any S, β with β satisfying Condition (1) and not necessarily (2). Suppose $H_\beta \neq \emptyset$ and $\beta(\beta - k) \geq 0$. Using Proposition 2.1, it is easy to see that [KT2, Sect. 3] continues to hold. Therefore $\mathcal{P}_{1-h+n, \beta}^{red}(S, [pt]^m[\gamma_1] \dots [\gamma_{2q(S)}])$ is given by a universal polynomial in $\beta^2, \beta.k, k^2, c_2(S)$ exactly as in [KT2, Thm. 1.1]. \oslash

Remark 4.5. The conditions for the duality formula of Theorem 1.2 are: $p_g(S) = 0$, $H_\beta, H_{k-\beta}$ both non-empty, and $\beta(\beta - k) \geq 0$. As we saw in the proof of Proposition 2.1 and Theorem 4.3, this implies $\beta(\beta - k) = 0$, $q(S) = 1$, and S is not a ruled surface or a blow-up of a ruled surface. Therefore S is hyper-elliptic, minimal properly elliptic, or a blow-up thereof. Conversely, any hyper-elliptic surface S or blow-up thereof and $\beta = k$ satisfies the conditions of Theorem 1.2. These examples are boring because $P_S^\pm(k) = 1$ (Example 3.3 and the blow-up formula [DKO, Thm. 3.12]). More exciting examples are provided by S as in Proposition 3.4 and $\beta = k$. From Theorem 4.3 it follows that these surfaces generally have $H_k \neq \emptyset$. Blowing up these surfaces, one obtains examples with $H_k \neq \emptyset$ and $k^2 \neq 0$. \oslash

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